Hermite Differential Equ..

Hermite Differential equ:

- The Equ is given by:

$$y'' - 2xy' + 2ny = 0 \quad ; \quad \text{where } n = \text{const} \neq 0$$

x=0 is a regular singular point.

Let the solution be: $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

Therefore, $y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$

and $y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$

Therefore,the equ becomes,

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} 2x\, a_\lambda (k+\lambda) x^{k+\lambda-1} + 2n \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

or, $\Sigma a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2\Sigma (k+\lambda-n)\, a_\lambda x^{k+\lambda} = 0$

Indicial Equation:

Equatting the coefficient of $x^{k+\lambda-2}$ (i.e lowest power of x term) with zero and putting $\lambda=0$,

$$\boxed{k(k-1)\,a_0 = 0} \quad \text{Indicial Equ.}$$

Therefore, $k=0$ & 1 [as $a_0 \neq 0$]

Recurrence Relation:

Equatting the coefficient of $x^{k+\lambda}$ (i.e highest power of x term) with zero ,

$$(k+\lambda+2)(k+\lambda+1)\,a_{\lambda+2} - 2(k+\lambda-n)\,a_\lambda = 0$$

$$a_{\lambda+2} = \frac{2(k+\lambda-n)\,a_\lambda}{(k+\lambda+2)(k+\lambda+1)}$$

This relation is called 'Recurrence Relation'.

For k=0

$$a_{\lambda+2} = \frac{2(\lambda-n)\;a_\lambda}{(\lambda+2)(\lambda+1)}$$

Solution for even λ :

for even values of λ,

$\lambda = 0, \quad a_2 = -\, 2n\, a_0 / 2.1$

$\lambda = 2, \quad a_4 = -\, 2(n-2)\, a_2 / 4.3 \;=\; 2^2 n(n-2)\, a_0 / 4!$

$\lambda = 4, \quad a_6 = -\, 2(n-4)\, a_4 / 6.5 \;=\; -2^3 n(n-2)(n-4)\, a_0 / 6!$

And so on....

The general term,

$$a_{2j} = \frac{(-2)^j\, n(n-2)(n-4)(n-6)\ldots\ldots(n-2j+2)a_0}{(2j)!}$$

Where j=1,2,3,4……….

Solution for odd λ :

for odd values of λ,

$\lambda = 1, \quad a_3 = -\,2(n-1)\, a_1 / 3.2$

$\lambda = 3, \quad a_5 = -\,2(n-3)\, a_3 / 5.4 \;=\; 2^2(n-1)(n-3)\, a_1 / 5!$

$\lambda = 5, \quad a_7 = -\,2(n-5)\, a_5 / 7.6 \;=\; -2^3(n-1)(n-3)(n-5)\, a_1 / 7!$

And so on....

The general term,

$$a_{2j+1} = \frac{(-2)^j (n-1)(n-3)(n-5)\ldots\ldots(n-2j+1)a_1}{(2j+1)!}$$

Where j=1,2,3,4..........

General Solution of Hermite Equ:

The general solution,

$$y(x) = a_0 \left[1 + \sum_{j=1}^{\infty} \frac{(-2)^j\, n(n-2)(n-4)(n-6)\ldots\ldots(n-2j+2)\, x^{2j}}{(2j)!} \right]$$

$$+ a_1 x \left[1 + \sum_{j=1}^{\infty} \frac{(-2)^j\, (n-1)(n-3)(n-5)\ldots\ldots(n-2j+1)\, x^{2j}}{(2j+1)!} \right]$$

Even series becomes finite even polynomial of degree ‘n’ when $(n-2j+2) = 0$

Odd series becomes finite even polynomial of degree ‘n’ when $(n-2j+1) = 0$

Formulation Of Hermite Polynomial (Even series)

To convert these series solution to a polynomial, both the series (Even & Odd) must be same.

Choose

$$a_0 = \frac{(-1)^{n/2}\; n!}{(n/2)!}$$

In even series,the term containing x^n (i.e $2j=n$)is given by,

$$y(x) = \frac{(-1)^{n/2} n!}{(n/2)!} \frac{(-2)^{n/2}\, n(n-2)(n-4)(n-6)\ldots\ldots(n-n+2)}{n!} x^n$$

$$= \frac{(-1)^{n/2}(-1)^{n/2}\, n! \;\; (2)^{n/2}\, (2)^{n/2}\, n/2(n/2-1)\ldots\ldots 1}{(n/2)! \qquad\qquad n!} x^n$$

$$= \frac{2^n\, n/2(n/2-1)\ldots\ldots 1}{(n/2)!} x^n$$

$$= (2x)^n$$